

April 24, 2008

Ruppeiner geometry and 2D dilaton gravity in the thermodynamics of black holes

Yun Soo Myung^{*} and Yong-Wan Kim[†]

*Institute of Basic Science and School of Computer Aided Science,
Inje University, Gimhae 621-749, Korea*

Young-Jai Park[‡]

*Department of Physics and Center for Quantum Spacetime,
Sogang University, Seoul 121-742, Korea*

Abstract

We study the geometric approach to the black hole thermodynamics. The geometric description of the equilibrium thermodynamics comes from Ruppeiner geometry based on a metric on the thermodynamic state space. For this purpose, we consider the Reissner-Nordström-AdS (RN-AdS) black hole which provides two different ensembles: canonical ensemble for fixed-charge case and grand canonical ensemble for fixed-potential case. Two cases are independent and cannot be mixed into each other. Hence, we calculate different Ruppeiner curvatures for two ensembles. However, we could not find the consistent behaviors of Ruppeiner curvature corresponding to those of heat capacity. Alternatively, we propose the curvature scalar in the 2D dilaton gravity approach which shows the features of extremal, Davies and minimum temperature points of RN-AdS black hole, clearly.

PACS numbers: 04.70.Dy, 04.60.Kz, 02.40.Ky

Keywords: Black hole thermodynamics, Ruppeiner curvature, 2D dilaton gravity

^{*} ysmyoung@inje.ac.kr

[†] ywkim65@gmail.com

[‡] yjpark@sogang.ac.kr

I. INTRODUCTION

Since the pioneering work of Bekenstein [1, 2, 3] and Hawking [4], which proved that the entropy of black hole is proportional to the area of its horizon in the early 1970s, the research of the black hole thermodynamics has greatly improved. Especially, it was found that if the surface gravity of the black hole is considered to be the temperature and the outer horizon area is considered to be the entropy, four laws of the black hole thermodynamics, which correspond with the four laws of the elementary thermodynamics, has been established [5]. Recently, there are several geometric approaches [6] to the black hole thermodynamics [7, 8, 9, 10, 11, 12]. The geometric description of the equilibrium thermodynamics comes from Ruppeiner geometry based on the metric of the thermodynamic state space: entropy metric and others including Weinhold metric [13]. However, the results depends on how to choose the metric. Very recently, Ruppeiner has discussed on a systematic discussion of how to make the correct choice of a metric [14]. Also, Ruppeiner has demonstrated several limiting results matching extremal Kerr-Newman black hole thermodynamics to the two dimensional Fermi gas [15], showing that this connection to a 2D model is consistent with the membrane paradigm of black holes [16].

On the other hand, 2D dilaton gravity has been used in various situations as an effective description of 4D gravity after a black hole in string theory has appeared [17, 18]. In particular, thermodynamics of this black hole has been analyzed by several authors [19, 20, 21, 22]. Another 2D theories, which were originated from the Jackiw-Teitelboim (JT) theory [23, 24], have been also studied [25, 26, 27]. Recently, we have introduced the 2D dilaton gravity approach, which completely preserves the thermodynamics of 4D black hole [28]. We have obtained that the 2D curvature scalar shows the features of extremal, Davies and minimum temperature points of RN black hole, clearly. Actually, since the heat capacity is inversely proportional to the 2D curvature scalar, we expect that the 2D curvature could show interesting thermodynamic points clearly.

In this paper, we deal with the issue of geometric approach by analyzing the RN-AdS black hole, which provides two different ensembles: canonical ensemble for fixed-charge case and grand canonical ensemble for fixed-potential case [29]. Since these are independent of each other, one cannot mix them by Legendre transformation. Hence, we have to calculate two Ruppeiner curvatures for different ensembles. In Sec. II, we briefly recapitulate the

RN-AdS black hole thermodynamics. In Sec. III, we calculate different Ruppeiner curvatures based on fixed-charge and fixed-potential ensembles. In Sec. IV, alternatively, we propose the 2D curvature scalar, which shows the features of extremal, Davies and minimum temperature points of the RN-AdS black hole, clearly. In Sec. V, we point out the different thermodynamic behaviors between Ruppeiner geometry and 2D dilaton gravity in the thermodynamics of black holes. Finally, discussions are devoted to Sec. VI.

II. RN-ADS BLACK HOLE THERMODYNAMICS

We start with the four-dimensional action

$$I_4 = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[R_4 - F_{\mu\nu} F^{\mu\nu} + \frac{6}{l^2} \right] \quad (1)$$

with l the curvature radius of AdS_4 spacetimes. Hereafter we shall adopt Planck units of $G = \hbar = c = k = 1$. The solutions to equations of motion lead to the RN-AdS black hole whose metric is given by

$$ds_{RN-AdS}^2 = -U(r)dt^2 + U^{-1}(r)dr^2 + r^2 d\Omega_2^2 \quad (2)$$

with the metric function $U(r)$

$$U(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}, \quad (3)$$

and $F_{\mu\nu} F^{\mu\nu} = -2Q^2/r^4$. Here, $d\Omega_2^2$ denotes $d\theta^2 + \sin^2\theta d\varphi^2$. We note that the reduced mass becomes the ADM mass ($m = 2M$) and the charge parameter becomes the charge ($q = Q$) when choosing Planck units. Then, the inner (r_-) and the outer (r_+) horizons are obtained from the condition of $U(r_{\pm}) = 0$. Using these horizons, the mass and charge are expressed by

$$M(r_+, r_-) = \frac{1}{2} \left[r_+ + r_- + \frac{r_+^4 - r_-^4}{l^2(r_+ - r_-)} \right], \quad Q^2(r_+, r_-) = r_+ r_- \left[1 + \frac{r_+^3 - r_-^3}{l^2(r_+ - r_-)} \right]. \quad (4)$$

First, let us consider the case of fixed-charge ensemble. For the RN-AdS black hole [29], the relevant thermodynamic quantities are given by the Bekenstein-Hawking entropy S , ADM mass M_Q , and Hawking temperature T_Q

$$S(r_+) = \pi r_+^2, \quad M_Q(r_+, Q) = \frac{r_+}{2} \left[1 + \frac{Q^2}{r_+^2} + \frac{r_+^2}{l^2} \right], \quad T_Q(r_+, Q) = \frac{1}{4\pi} \left[\frac{1}{r_+} - \frac{Q^2}{r_+^3} + \frac{3r_+}{l^2} \right]. \quad (5)$$

In this case that the horizon is degenerate ($r_+ = r_- = r_e$), we have an extremal black hole with $M = M_e = r_e$. In general, one has an inequality of $M > M_e$. Then, using the Eq. (5), the heat capacity $C_Q = (dM_Q/dT_Q)_Q$ for fixed-charge Q takes the form

$$C_Q(r_+, Q) = 2\pi r_+^2 \left[\frac{3r_+^4 + l^2(r_+^2 - Q^2)}{3r_+^4 + l^2(-r_+^2 + 3Q^2)} \right]. \quad (6)$$

The global features of thermodynamic quantities are shown in Fig. 1 for $Q < Q_c = l/6$. Here we observe the local minimum $T_Q = T_0$ at $r_+ = r_0$ (Schwarzschild-AdS black hole), in addition to the zero temperature $T_H = 0$ at $r_+ = r_e$ (extremal RN black hole) and the local maximum $T_Q = T_D$ at $r_+ = r_D$ (Davies' point of RN black hole). It seems to be a combination of the RN and Schwarzschild-AdS black holes.

Here, we observe that $C_Q = 0$ and $T_Q = 0$ at $r_+ = r_e$,

$$r_e^2 = \frac{l^2}{6} \left[-1 + \sqrt{1 + \frac{12Q^2}{l^2}} \right]. \quad (7)$$

and the heat capacity blows up at $r_+ = r_D$ and r_0 where these satisfy

$$r_D^2 = \frac{l^2}{6} \left[1 - \sqrt{1 - \frac{36Q^2}{l^2}} \right], \quad r_0^2 = \frac{l^2}{6} \left[1 + \sqrt{1 - \frac{36Q^2}{l^2}} \right]. \quad (8)$$

These points exist only for $Q < Q_c = l/6$. For $Q = Q_c$, we have $r_D = r_0 = l/\sqrt{6}$. Here we consider the thermal stability of a black hole. The local stability is usually determined by the positive sign of heat capacity by considering evaporation and absorbing processes of a black hole, even for an exceptional case of the Kerr-Newman black hole [30]. For example, the heat capacity of the Schwarzschild black hole is $-2\pi r_+^2$, which means that this isolated black hole is not in equilibrium in asymptotically flat spacetimes. Based on the local stability of heat capacity, the RN-AdS black holes of $Q < Q_c$ can be split into small stable black hole with $C_Q > 0$ being in the region of $r_e < r_+ < r_D$, intermediate unstable black hole with $C_Q < 0$ in the region of $r_D < r_+ < r_0$, and large stable black hole with $C_Q > 0$ in the region of $r_+ > r_0$.

On the other hand, for fixed-potential $\Phi = Q/r_+$ ensemble [29], we have the mass and Hawking temperature

$$M_\Phi(r_+, \Phi) = \frac{r_+}{2} \left[1 + \Phi^2 + \frac{r_+^2}{l^2} \right], \quad T_\Phi(r_+, \Phi) = \frac{1}{4\pi r_+} \left[1 - \Phi^2 + \frac{3r_+^2}{l^2} \right]. \quad (9)$$

In this ensemble, a relevant thermodynamic quantity is the internal energy J_Φ defined by

$$J_\Phi(r_+, \Phi) = M_\Phi - \Phi Q = \frac{r_+}{2} \left[1 - \Phi^2 + \frac{r_+^2}{l^2} \right]. \quad (10)$$

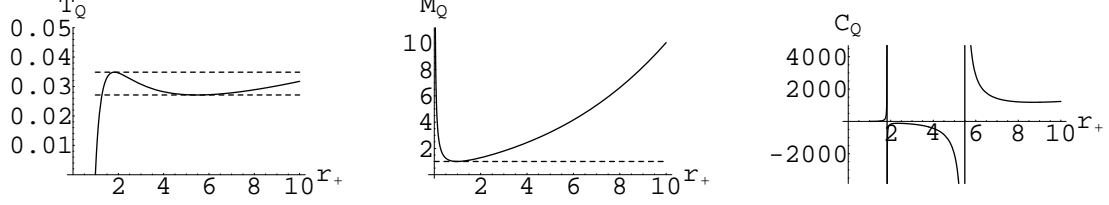


FIG. 1: Thermodynamic quantities of the RN-AdS black hole as function of horizon radius r_+ with fixed $Q = 1 < Q_c$ and $l = 10$: temperature T_Q with $T = T_D = 0.035, T_0 = 0.027$ (dashed lines), mass M_Q with $M = M_e = 1.005$ (dashed line) and heat capacity C_Q .

Furthermore, the heat capacity $C_\Phi = T_\Phi(\partial S/\partial T_\Phi)_\Phi$ takes the form

$$C_\Phi(r_+, \Phi) = 2\pi r_+^2 \left[\frac{3r_+^2 + l^2(1 - \Phi^2)}{3r_+^2 + l^2(-1 + 3\Phi^2)} \right]. \quad (11)$$

In this grand canonical ensemble, the points $r_+ = r_e, r_D, r_0$ of zero-temperature and the blow-up heat capacity are not those in the canonical ensemble [31]. Furthermore, if we compare Fig. 2 for $Q < Q_c$ with Fig. 3 for $\Phi < \Phi_c$, we could find that C_Φ displays no singular behavior at the singular points $r_+ = r_D, r_0$ of C_Q . Hence we must separate the fixed-potential case from the fixed-charge one.

III. RUPPEINER CURVATURE

A. (S, Q) -representation

Another observation on the geometric description of the equilibrium thermodynamics comes from Ruppeiner geometry based on a metric on the thermodynamic state space [8]

$$ds_R^2 = g_{ij}^R dx^i dx^j = \frac{1}{T_Q} g_{ij}^W d\tilde{x}^i d\tilde{x}^j, \quad x^i = (M, Q), \quad \tilde{x}^i = (S, Q), \quad (12)$$

where the Ruppeiner metric g_{ij}^R and the Weinhold metric g_{ij}^W are given by

$$g_{ij}^R = -\frac{\partial^2 S(x^i)}{\partial x^i \partial x^j}, \quad g_{ij}^W = \frac{\partial^2 M_Q(\tilde{x}^i)}{\partial \tilde{x}^i \partial \tilde{x}^j}. \quad (13)$$

In this case, we use the Ruppeiner metric as

$$g_{ij}^Q = \frac{1}{T_Q} \frac{\partial^2 M_Q}{\partial \tilde{x}^i \partial \tilde{x}^j} \quad (14)$$

whose form is given by

$$g_{ij}^Q = \frac{1}{2S(1 - \frac{\pi Q^2}{S} + \frac{3S}{\pi l^2})} \begin{pmatrix} -1 + \frac{3\pi Q^2}{S} + \frac{3S}{\pi l^2} & 4\pi Q \\ 4\pi Q & 8\pi S \end{pmatrix} \quad (15)$$

We observe the important relation

$$g_{SS}^Q = \frac{1}{T_Q} \left(\frac{\partial^2 M_Q}{\partial S^2} \right)_Q = \frac{1}{C_Q(S, Q)}. \quad (16)$$

Its determinant takes the form

$$\text{Det}[g_{ij}] = -\frac{2\pi}{S} \frac{(1 - \frac{\pi Q^2}{S} - \frac{3S}{\pi l^2})}{(1 - \frac{\pi Q^2}{S} + \frac{3S}{\pi l^2})^2} \quad (17)$$

One Ruppeiner curvature is given by [9]

$$\mathcal{R}_Q^I = \frac{9}{\pi l^2} \frac{(1 - \frac{\pi Q^2}{S} - \frac{S}{\pi l^2})(\frac{\pi Q^2}{S} + \frac{3S}{\pi l^2})}{(1 - \frac{\pi Q^2}{S} + \frac{3S}{\pi l^2})(1 - \frac{\pi Q^2}{S} - \frac{3S}{\pi l^2})^2}. \quad (18)$$

In this work, we use the computer program *Mathematica* 6.0 and *General Relativity, Einstein & All* package (GREAT.m) [32] for the calculations of all Ruppeiner curvatures. We observe that the curvature diverges in the extremal limit of $1 - \frac{\pi Q^2}{S} + \frac{3S}{\pi l^2} = 0$ ($r_+ = r_-$). However, the two blow-up points of heat capacity which satisfy $1 - \frac{3\pi Q^2}{S} - \frac{3S}{\pi l^2} = 0$ ($3r_+^4 - l^2 r_+^2 + 3l^2 Q^2 = 0$) disappear in the denominator of \mathcal{R}_Q^I . This arises because off-diagonal terms in g_{ij}^Q contribute to calculating $\text{Det}[g_{ij}]$. Dimensional reduction from KN-AdS black hole (when $J \rightarrow 0$) provides the other Ruppeiner curvature [12]

$$\mathcal{R}_Q^{II} \propto -\frac{1}{(\pi l^2)^7 S^6 (1 + \frac{S}{\pi l^2})^2 (1 + \frac{\pi Q^2}{S} + \frac{S}{\pi l^2})^2 (1 - \frac{\pi Q^2}{S} + \frac{3S}{\pi l^2}) (1 - \frac{\pi Q^2}{S} - \frac{3S}{\pi l^2})^2}, \quad (19)$$

which shows that this curvature diverges at the extremal point but is finite at $r_+ = r_D, r_0$.

B. (S, Φ) -representation

In this case, we use the Ruppeiner metric as

$$g_{ij}^\Phi = \frac{1}{T_\Phi} \frac{\partial^2 J_\Phi(S, \Phi)}{\partial \tilde{x}^i \partial \tilde{x}^j} \quad (20)$$

whose form is given by

$$g_{ij}^\Phi = \frac{1}{2S(1 - \Phi^2 + \frac{3S}{\pi l^2})} \begin{pmatrix} -1 + \Phi^2 + \frac{3S}{\pi l^2} & -4\Phi S \\ -4\Phi S & -8S^2 \end{pmatrix}. \quad (21)$$

Here, we use $J_\Phi = M_\Phi - \Phi Q$ instead of M_Φ because the former could be defined in the potential representation [11]. Here, we call J_Φ as a free energy since the Gibbs potential is defined by $G = J_\Phi - T_\Phi S$ in the grand canonical ensemble.

We note that $T_Q = T_\Phi$, but they have different representations. We observe the important relation

$$g_{SS}^\Phi = \frac{1}{T_\Phi} \left(\frac{\partial^2 J_\Phi}{\partial S^2} \right)_\Phi = \frac{1}{C_\Phi(S, \Phi)}. \quad (22)$$

Its determinant takes the form

$$\text{Det}[g_{ij}] = 2 \left(\frac{1 - 3\Phi^2 - \frac{3S}{\pi l^2}}{1 - \Phi^2 + \frac{3S}{\pi l^2}} \right). \quad (23)$$

Then, Ruppeiner curvature is given by

$$\mathcal{R}_\Phi = - \frac{(1 - \Phi^2)^2 - \frac{18S^2}{\pi^2 l^4} (1 - 3\Phi^2) + \frac{3S}{\pi^2 l^4} (3 - 9\Phi^2 + 10\Phi^4)}{S(1 - \Phi^2 + \frac{3S}{\pi l^2})(1 - 3\Phi^2 - \frac{3S}{\pi l^2})^2}. \quad (24)$$

We observe that the curvature diverges at the extremal black hole of $S = S_e$ which satisfies $1 - \Phi^2 + \frac{3S}{\pi l^2} = 0$, existing for $\Phi \geq \Phi_c$.

Finally, we comment that it is meaningless to get on-diagonal form by starting from g_{ij}^Q in Eq. (15) and then conformally transforming with a new coordinate of $u = \Phi$ [9]. The resulting metric is just the diagonal part of g_{ij}^Φ in Eq. (21). This is because it gives rise to mixing between fixed-charge g_{ij}^Q and fixed-potential g_{ij}^Φ .

IV. 2D DILATON GRAVITY INDUCED BY DIMENSIONAL REDUCTION

Assuming $\mathcal{M}_4 = \mathcal{M}_2 \times S^2$ for our purpose, we perform a Kaluza-Klein reduction

$$ds_{RN-AdS}^2 = h_{\alpha\beta}(t, r) dx^\alpha dx^\beta + b^2 d\Omega_2^2, \quad (25)$$

where b represents the radius of two sphere S^2 . After the dimensional reduction by integrating Eq.(1) over S^2 , the 2D action is given by

$$I_2 = \frac{1}{4} \int_{\mathcal{M}_2} dr dt \sqrt{-h} \left[b^2 R_2(h) + 2h^{\alpha\beta} \nabla_\alpha b \nabla_\beta b + 2 - b^2 \left(F_{\alpha\beta} F^{\alpha\beta} - \frac{6}{l^2} \right) \right]. \quad (26)$$

Here $R_2(h)$ is the 2D Ricci scalar. For thermodynamic analysis, it is convenient to eliminate the kinetic term by using the conformal transformation [33, 34, 35]

$$\bar{h}_{\alpha\beta} = \sqrt{\phi} h_{\alpha\beta}, \quad \phi = \frac{b^2}{4}. \quad (27)$$

The 2D action is given by

$$\bar{I}_2 = \int_{\bar{\mathcal{M}}_2} dx dt \sqrt{-\bar{h}} \left[\phi \bar{R}_2 + \frac{1}{2\sqrt{\phi}} - \phi^{3/2} \bar{F}_{\alpha\beta} \bar{F}^{\alpha\beta} + \frac{6\sqrt{\phi}}{l^2} \right]. \quad (28)$$

where \bar{R} is the 2D Ricci scalar in the conformal transformed frame. This transformation delivers information on the 4D action (1) to the 2D dilaton potential completely, if the 4D action provides the black hole solution [28]. That is, we may get the good s -wave approximation to the 4D black hole without kinetic term for thermodynamic analysis. Unless one makes the conformal transformation, the information is split into the kinetic and the potential terms [36].

Making use of equation of motion for the electromagnetic field yields to [37]

$$\frac{\phi^{3/2} \bar{F}_{+-}}{\sqrt{-\bar{h}}} = \bar{Q}, \quad (29)$$

where \bar{Q} is an integration constant. In order to recover the RN-AdS black hole, we choose $\bar{Q} = Q/4$ so that

$$\bar{F}_{\alpha\beta} \bar{F}^{\alpha\beta} = 2\bar{F}_{+-}^2 = \frac{Q^2}{8\phi^3}. \quad (30)$$

The 2D effective action is obtained as

$$\bar{I}_2 = \int_{\bar{\mathcal{M}}_2} dx dt \sqrt{-\bar{h}} [\phi \bar{R}_2 + V_Q]. \quad (31)$$

For the fixed charge ensemble, the dilaton potential is given by

$$V_Q(\phi, Q) = \frac{1}{2\sqrt{\phi}} - \frac{Q^2}{8\phi^{3/2}} + \frac{6\sqrt{\phi}}{l^2}. \quad (32)$$

This is an effective 2D dilaton gravity. The two equations of motion are

$$\bar{\nabla}^2 \phi = V_Q(\phi), \quad (33)$$

$$\bar{R}_Q = -\frac{dV_Q}{d\phi}, \quad (34)$$

where the derivative of V is given by

$$\frac{dV_Q}{d\phi} = -\frac{1}{4\phi^{3/2}} + \frac{3Q^2}{16\phi^{5/2}} + \frac{3}{l^2\sqrt{\phi}}. \quad (35)$$

The former is called the dilaton equation and the latter is the Einstein equation, even though the former(latter) are obtained from metric(dilaton) variations. In order to solve the above equations, we introduce the Schwarzschild-type metric for $\bar{h}_{\alpha\beta}$ as

$$\bar{h}_{\alpha\beta} = \text{diag}(-f, f^{-1}). \quad (36)$$

Then, its curvature scalar takes the form

$$\bar{R}_Q = -f'', \quad (37)$$

where the prime “ \prime ” denotes the derivative with respect to x . We note from Eq.(27) that the dilaton is independent of time $t(\phi = \phi(x))$. Then, Eqs. (33) and (34) reduce to

$$f\phi'' + f'\phi' = V_Q(\phi), \quad f'' = \frac{dV_Q}{d\phi}. \quad (38)$$

In addition, we have the kinetic term for ϕ

$$(\bar{\nabla}\phi)^2 = f(\phi')^2. \quad (39)$$

If one chooses the linear dilaton as the solution

$$\phi = x, \quad (40)$$

then Eq.(38) leads to

$$f' = V_Q, \quad f'' = V'_Q, \quad (41)$$

which implies that the latter is just a redundant relation. We obtain the solution to Eqs. (33) and (34)

$$ds_{2D}^2 = \bar{h}_{\alpha\beta} dx^\alpha dx^\beta = -f(\phi, Q) dt^2 + \frac{dx^2}{f(\phi, Q)}. \quad (42)$$

Here, the metric function $f(\phi, Q)$ is given by

$$f(\phi, Q) = J_Q(\phi, Q) - \mathcal{C}, \quad (43)$$

where $J_Q(\phi, Q)$ is the integration of V_Q given by

$$J_Q(\phi, Q) = \int^\phi V_Q(\tilde{\phi}, Q) d\tilde{\phi} = \sqrt{\phi} + \frac{Q^2}{4\sqrt{\phi}} + \frac{4\phi\sqrt{\phi}}{l^2}. \quad (44)$$

Also, \mathcal{C} is a coordinate-invariant constant of integration, which is identified with the mass M_Q of the RN-AdS black hole. We note here an important connection between $J_Q(\phi)$ and the metric function $U(\phi)$: $J_Q(\phi) - M = \sqrt{\phi}U(\phi)$. Then, we recover the t - r line element for RN-AdS black hole from the 2D dilaton gravity approach with $\phi = x = r^2/4$

$$ds_{2D}^2 = \sqrt{\phi} h_{\alpha\beta} dx^\alpha dx^\beta = \sqrt{\phi} \left[-U(r) dt^2 + \frac{dr^2}{U(r)} \right] = \frac{r}{2} ds_{RN-AdS}^2. \quad (45)$$

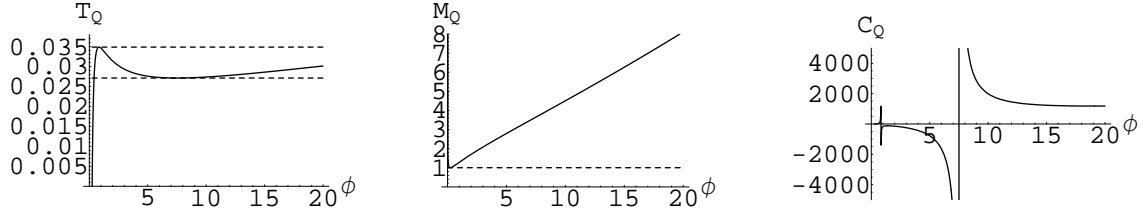


FIG. 2: Three graphs for the 2D dilaton gravity with $Q = 1$. (a) The solid curve describes T_Q with two dashed horizons for $T_Q = T_D (= 0.035)$, $T_0 (= 0.027)$ at $\phi = \phi_D (= 0.83)$, $\phi_0 (= 7.5)$, respectively. (b) Mass M_Q whose minimum is $M_e = 1.005$ at $\phi = \phi_e (= 0.24)$. (c) Heat capacity C_Q which diverges at $\phi = \phi_D$, ϕ_0 .

We note that the outer and inner horizons appear when satisfying

$$f(\phi_{\pm}) = 0 \rightarrow U(r_{\pm}) = 0. \quad (46)$$

In this case we find from Eq. (39) that $(\nabla\phi)^2 = 0$, even for $\phi' \neq 0$. Note that if one starts from Eq. (25) without the conformal transformation, it is tedious to find the solution of $h_{\alpha\beta} = \text{diag}[-U(r), U(r)^{-1}]$ by solving the full Einstein and dilation equations. Hence the conformal transformation of Eq. (27) provides an efficient way to find the 4D black hole solution when using the 2D dilaton gravity. Further, we could easily recover 4D thermodynamic quantities when using the conformal transformation. Given that the considering 4D model admits black hole solution, it can be shown that all thermodynamic quantities of the black hole except the free energy are invariant under the conformal transformations.

All thermodynamic quantities can be explicitly expressed in terms of the dilaton ϕ , the dilaton potential $V_Q(\phi, Q)$, its integration $J_Q(\phi, Q)$, and its derivative $V'_Q(\phi, Q)$ with respect to ϕ as

$$S = 4\pi\phi, \quad M_Q(\phi, Q) = J_Q(\phi, Q), \quad T_Q(\phi, Q) = \frac{V_Q(\phi, Q)}{4\pi}, \quad C_Q(\phi, Q) = 4\pi \frac{V_Q(\phi, Q)}{V'_Q(\phi, Q)}. \quad (47)$$

In Fig. 2, we have the corresponding dual graphs, which are nearly the same as in Fig. 1. At the extremal point $\phi = \phi_e$, we have $T_Q = 0$ and $C_Q = 0$, which are determined by $V_Q(\phi_e) = 0$. On the other hand, at the local maximum point $\phi = \phi_D$, one has $T_Q = T_D$, $C_Q = \pm\infty$, which are determined by the condition of $V'_Q(\phi_D) = 0$. Finally, at the local minimum point $\phi = \phi_0$, one has $T_Q = T_0$, $C_Q = \pm\infty$, which are determined by the condition of $V'_Q(\phi_0) = 0$. For $\phi_e < \phi < \phi_D$, we have the near-horizon phase of extremal black hole, whereas for $\phi > \phi_D$, we have the Schwarzschild-AdS phase.

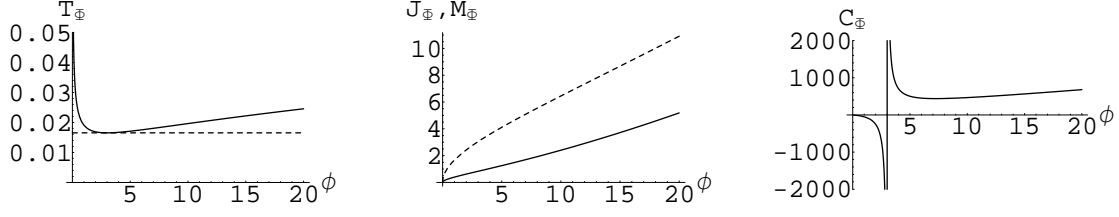


FIG. 3: Three graphs for the 2D dilaton gravity with $\Phi = 0.8 < \Phi_c$. (a) The solid curve describes T_Φ with dashed horizon for the minimum temperature $T = T_0 (= 0.017)$ at $\phi = \tilde{\phi}_0 (= 3)$. (b) Internal energy J_Φ (solid curve) and mass M_Φ (dashed curve). (c) Heat capacity C_Φ which diverges at $\phi = \tilde{\phi}_0$.

On the other hand, for the fixed-potential ensemble, we have a slight different form

$$V_\Phi(\phi, \Phi) = \frac{1}{2\sqrt{\phi}}[1 - \Phi^2] + \frac{6\sqrt{\phi}}{l^2}. \quad (48)$$

Its derivative with respect to ϕ takes the form

$$\frac{dV_\Phi(\phi, \Phi)}{d\phi} = \frac{1}{4\phi^{3/2}}[-1 + \Phi^2] + \frac{3}{l^2\sqrt{\phi}}. \quad (49)$$

Its integration over ϕ leads to the internal energy

$$J_\Phi(\phi, \Phi) = \sqrt{\phi}[1 - \Phi^2] + \frac{4\phi\sqrt{\phi}}{l^2}. \quad (50)$$

However, the ADM mass is given by

$$M_\Phi(\phi, \Phi) = J_\Phi + \Phi Q = \sqrt{\phi}[1 + \Phi^2] + \frac{4\phi\sqrt{\phi}}{l^2}. \quad (51)$$

Other thermodynamic quantities are shown to have

$$T_\Phi(\phi, \Phi) = \frac{V_\Phi(\phi, \Phi)}{4\pi}, \quad C_\Phi(\phi, \Phi) = T_\Phi \left(\frac{\partial S}{\partial T_\Phi} \right)_\Phi = 4\pi \frac{V_\Phi(\phi, \Phi)}{V'_\Phi(\phi, \Phi)}. \quad (52)$$

For $\Phi < \Phi_c = 1$, their behaviors are depicted in Fig. 3. In this case, we have no degenerate horizon but a blow-up heat capacity at $\phi = \tilde{\phi}_0$. On the other hand, for $\Phi \geq \Phi_c = 1$, we have degenerate horizon which satisfies $V_\Phi = 0$ but not a blow-up heat capacity. Also, Eq. (34) takes the form

$$\bar{R}_\Phi = -\frac{dV_\Phi(\phi, \Phi)}{d\phi}. \quad (53)$$

V. THERMODYNAMIC RELATIONS

A. Fixed-charge ensemble

First, let us consider the fixed-charge Q ensemble. Using the relation of $r_+ = \sqrt{S/\pi}$ together with Eqs. (5) and (6) or $\phi = S/4\pi$ with Eq. (47), we can rewrite thermodynamic quantities as functions of the entropy S and charge Q

$$M_Q(S, Q) = \frac{1}{2} \sqrt{\frac{S}{\pi}} \left(1 + \frac{\pi Q^2}{S} + \frac{S}{\pi l^2} \right) \quad (54)$$

$$T_Q(S, Q) = \frac{1}{4\sqrt{\pi S}} \left(1 - \frac{\pi Q^2}{S} + \frac{3S}{\pi l^2} \right), \quad (55)$$

$$C_Q(S, Q) = 2S \left[\frac{1 - \frac{\pi Q^2}{S} + \frac{3S}{\pi l^2}}{-1 + \frac{3\pi Q^2}{S} + \frac{3S}{\pi l^2}} \right]. \quad (56)$$

We introduce the first-law of the thermodynamics

$$dM_Q = T_Q dS + \Phi dQ \quad (57)$$

with the chemical potential Φ . We have the thermodynamic relations

$$T_Q = \left(\frac{\partial M_Q}{\partial S} \right)_Q, \quad \left(\frac{\partial T_Q}{\partial S} \right)_Q = \left(\frac{\partial^2 M_Q}{\partial S^2} \right)_Q. \quad (58)$$

From Eqs.(34) and (35), the 2D curvature scalar \bar{R}_Q defined by Eq.(37) takes the form

$$\bar{R}_Q(S, Q) = -(4\pi)^2 \left(\frac{\partial T_Q}{\partial S} \right)_Q = -\frac{2\pi^{3/2}}{S\sqrt{S}} \left(-1 + \frac{3\pi Q^2}{S} + \frac{3S}{\pi l^2} \right), \quad (59)$$

which shows that the 2D curvature constructed from Eq.(36) could describe thermodynamics of RN-AdS black hole when using the 2D Einstein equation. Importantly, the heat capacity can be rewritten as

$$C_Q(S, Q) = T_Q \left(\frac{\partial T_Q}{\partial S} \right)_Q^{-1} = -(4\pi)^2 \frac{T_Q}{\bar{R}_Q}, \quad (60)$$

which shows clearly that C_Q blows up when $\bar{R}_Q = 0$, and it is zero when $T_Q = 0$. As is shown in Fig. 4, the 2D curvature shows that $\bar{R}_Q = -4.42$ (AdS₂ spacetime) at the extremal point of $\phi = \phi_e$, while it is zero (flat spacetime) at the Davies' point $\phi = \phi_D$ and minimum temperature point $\phi = \phi_0$. This shows a close connection between the heat capacity and 2D curvature in the canonical ensemble.

We note one connection such as

$$1 - \frac{\pi Q^2}{S} + \frac{3S}{\pi l^2} = 0 \longleftrightarrow l^2 r_+^2 - l^2 Q^2 + 3r_+^4 = 0 \quad (61)$$

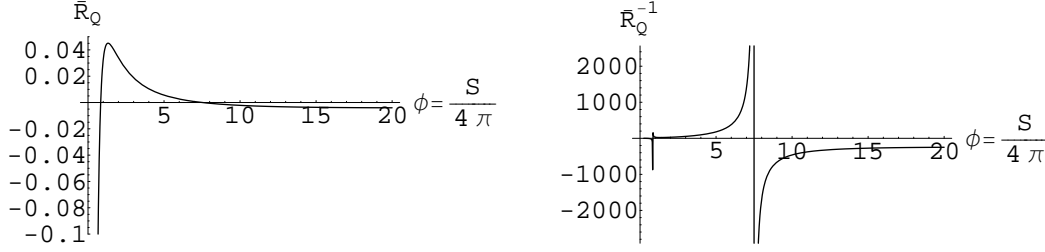


FIG. 4: Graphs for 2D curvature \bar{R}_Q and its inverse curvature \bar{R}_Q^{-1} as functions of $\phi = S/4\pi$ for $Q = 1 < Q_c$. We find that $\bar{R}_Q = -4.42, 0, 0$ for $\phi = \phi_e, \phi_D, \phi_0$, respectively. We have AdS₂, flat, dS₂ spacetimes, depending on the value of $\bar{R}_Q < 0, = 0, > 0$. On the other hand, \bar{R}_Q^{-1} blows up at $\phi = \phi_D, \phi_0$, showing similar behavior of heat capacity C_Q .

whose solution is $r_+^2 = r_e^2$, $S = S_e = \pi r_e^2$. We also have another connection,

$$-1 + \frac{3\pi Q^2}{S} + \frac{3S}{\pi l^2} = 0 \longleftrightarrow -l^2 r_+^2 + 3l^2 Q^2 + 3r_+^4 = 0 \quad (62)$$

whose solution is $r_+^2 = r_D^2$, $S = S_D = \pi r_D^2$ and $r_+^2 = r_0^2$, $S = S_0 = \pi r_0^2$.

B. Fixed-potential ensemble

Next, let us consider the fixed-potential ensemble. We can rewrite thermodynamic quantities as functions of the entropy S and the chemical potential Φ

$$J_\Phi(S, \Phi) = M_\Phi - \Phi Q = \frac{1}{2} \sqrt{\frac{S}{\pi}} \left(1 - \Phi^2 + \frac{S}{\pi l^2} \right), \quad (63)$$

$$T_\Phi(S, \Phi) = \frac{1}{4\sqrt{\pi S}} \left(1 - \Phi^2 + \frac{3S}{\pi l^2} \right), \quad (64)$$

$$C_\Phi(S, \Phi) = 2S \left[\frac{1 - \Phi^2 + \frac{3S}{\pi l^2}}{-1 + \Phi^2 + \frac{3S}{\pi l^2}} \right], \quad (65)$$

where

$$M_\Phi(S, \Phi) = \frac{1}{2} \sqrt{\frac{S}{\pi}} \left(1 + \Phi^2 + \frac{S}{\pi l^2} \right). \quad (66)$$

For $\Phi > \Phi_c = 1$, we could find zero temperature which implies the presence of extremal black hole at $S = S_e = \frac{\pi l^2}{3} [\Phi^2 - 1]$. Also, in the case of $\Phi = \Phi_c$, we could find zero temperature at $S = 0$. For $\Phi < \Phi_c$, we could not find zero temperature for any S . However, we find that

the heat capacity blows up for $\Phi < \Phi_c$ at $S = S_0 = \frac{\pi l^2}{3}[1 - \Phi^2]$. In this case, we expect that the RN-AdS black hole shows a similar behavior as the Schwarzschild-AdS black hole.

Now, let us introduce the law of the thermodynamics for the $J_\Phi(S, \Phi)$

$$dJ_\Phi = T_\Phi dS - Q d\Phi \quad (67)$$

with $dM_\Phi = T_\Phi dS + Q d\Phi$. Also note that the redefined mass J_Φ is identified with a free energy of the black hole because the Gibbs free energy is defined as $G = J_\Phi - T_\Phi S$. We note that the second term represents the work done by the black hole on its environment. Then, for fixed-charge Φ ensemble, we have the thermodynamic relations

$$T_\Phi = \left(\frac{\partial J_\Phi}{\partial S}\right)_\Phi, \quad \left(\frac{\partial T_\Phi}{\partial S}\right)_\Phi = \left(\frac{\partial^2 J_\Phi}{\partial S^2}\right)_\Phi. \quad (68)$$

From Eq. (34), the 2D curvature scalar \bar{R}_Φ takes the form

$$\bar{R}_\Phi(S, \Phi) = -(4\pi)^2 \left(\frac{\partial^2 J_\Phi}{\partial S^2}\right)_\Phi = -\frac{2\pi^{3/2}}{S\sqrt{S}} \left(-1 + \Phi^2 + \frac{3S}{\pi l^2}\right). \quad (69)$$

On the other hand, the heat capacity can be rewritten as

$$C_\Phi(S, \Phi) = T_\Phi \left(\frac{\partial T_\Phi}{\partial S}\right)_\Phi^{-1} = -(4\pi)^2 \frac{T_\Phi}{\bar{R}_\Phi}, \quad (70)$$

which shows clearly that C_Φ blows up when $\bar{R}_\Phi = 0$. As is shown in Fig. 5, the 2D curvature shows that it is zero (flat spacetime) at the minimum temperature point $\phi = \tilde{\phi}_0$. Similar to the canonical ensemble, this also shows a close connection between the heat capacity and 2D curvature in the grand canonical ensemble.

VI. DISCUSSIONS

We summarize the features of heat capacity, Ruppeiner curvature, and inverse 2D curvature at the extremal, Davies' and minimum temperature points in the (S, Q) -representation in Table I. We do not find any consistent relations between the heat capacity and four Ruppeiner curvatures. We note that $\mathcal{R} \propto 1/(\text{Det}[g_{ij}])^2$. If the Ruppeiner metric is not on-diagonal form, like Eq. (15), then the pole of \mathcal{R} is shifted from $1/(g_{SS}g_{QQ})^2 = (C_Q/g_{QQ})^2 \rightarrow 1/(g_{SS}g_{QQ} - g_{SQ}^2)^2$: $(1 - \frac{3\pi Q^2}{S} - \frac{3S}{\pi l^2})^2 = 0 \rightarrow (1 - \frac{\pi Q^2}{S} - \frac{3S}{\pi l^2})^2 = 0$. Hence the Ruppeiner curvature does not show the feature of heat capacity at the Davies' and minimum temperature

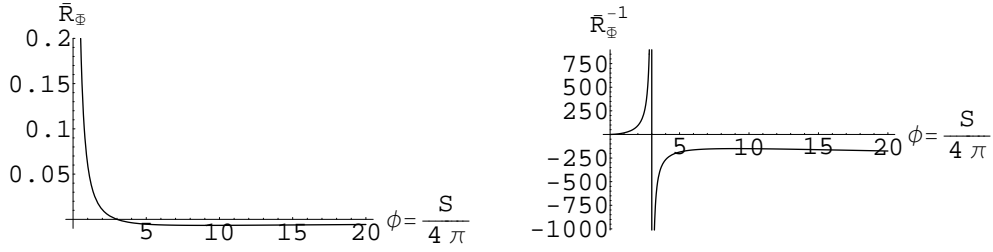


FIG. 5: Graphs for 2D curvature \bar{R}_Φ and its inverse curvature \bar{R}_Φ^{-1} as functions of $\phi = S/4\pi$ for $\Phi = 0.8 < \Phi_c$. We find that $\bar{R}_\Phi = 0$ for $\phi = \tilde{\phi}_0$. We have dS₂, flat, and AdS₂ spacetimes, depending on the value of $\bar{R}_\Phi > 0$, $= 0$, < 0 , respectively. On the other hand, \bar{R}_Φ^{-1} blows up at $\phi = \tilde{\phi}_0$, showing similar behavior of heat capacity C_Φ .

points where heat capacity blows up. However, if the metric is on-diagonal ($g_{SQ} = 0$), the Ruppeiner curvature preserves the pole of heat capacity as $1/g_{SS}^2 = C_Q^2$ is shown.

In the (S, Φ) -representation, the same happens. Since its Ruppeiner metric of Eq. (21) is not on-diagonal form, the pole of \mathcal{R} is shifted from $1/(g_{SS}g_{\Phi\Phi})^2 = (C_\Phi/g_{\Phi\Phi})^2 \rightarrow 1/(g_{SS}g_{\Phi\Phi} - g_{S\Phi}^2)^2$: $(1 - \Phi^2 - \frac{3S}{\pi l^2})^2 = 0 \rightarrow (1 - 3\Phi^2 - \frac{3S}{\pi l^2})^2 = 0$. Hence the Ruppeiner curvature does not preserve the feature of heat capacity at minimum temperature point where heat capacity blows up.

At this stage, we comment on the connection between Ruppeiner geometric and thermodynamic ensemble approaches. In the canonical ensemble of fixed-charge, we have $g_{ij}^Q = \text{diag}[g_{SS}^Q, 0]$, while in the grand canonical ensemble of fixed-potential, we have $g_{ij}^\Phi = \text{diag}[g_{SS}^\Phi, 0]$. A single component of $g_{SS}^{Q/\Phi}$ contains the whole information on heat capacity of RN-AdS black hole. These two cases lead to flat thermodynamic space of $\mathcal{R}^{Q/\Phi} = 0$. It seems that Ruppeiner geometric approach is not compatible with the conventional thermodynamic ensemble approach of black holes.

Although we have performed for the RN-AdS black hole, this behavior persists to any black holes. We have shown that all thermodynamic quantities of the RN-AdS black holes can be described from the 2D dilaton gravity obtained by performing the dimensional reduction and conformal transformation. Also it is known that entropy, temperature, mass and specific heat are invariant under the conformal transformations.

Hence, as a consistent connection to the membrane paradigm of black holes, we propose

	Extremal point(RN)	Davies' point(RN)	Minimum point(SAdS)	Remarks
C_Q	0	diverges	diverges	Heat capacity
\mathcal{R}_Q^I	diverges	finite	finite	RC
\mathcal{R}_Q^{II}	diverges	finite	finite	RC
\bar{R}_2^{-1}	finite	diverges	diverges	2D curvature

TABLE I: Heat capacity C_Q , Ruppeiner curvatures \mathcal{R}_Q^I in[9] and \mathcal{R}_Q^{II} in[12], and inverse 2D curvature \bar{R}_2^{-1} for extremal, Davies, and minimum temperature points of RN-AdS black hole for $Q < Q_c$. RN (SAdS) denote the Reissner-Norström (Schwarzschild-AdS) black holes.

the promising 2D curvature, which carries out the information on the heat capacity. The inverse 2D curvature could well describe Davies' and minimum temperature points, as the heat capacity does.

Finally, considering Eqs.(59) and(69), we confirm that the negativity of \bar{R} (positivity of $\partial^2 M/\partial S^2$) is sufficient to ensure the local stability of RN-AdS black holes [38].

Acknowledgement

The authors thank Joey Medved for his helpful discussions. Two of us (Y.S. Myung and Y.-J. Park) were supported by the Science Research Center Program of the Korea Science and Engineering Foundation through the Center for Quantum Spacetime of Sogang University with grant number R11-2005-021. Y.-W. Kim was supported by the Korea Research Foundation Grant funded by Korea Government (MOEHRD): KRF-2007-359-C00007.

-
- [1] J. D. Bekenstein, Lett. Nuovo Cimento 4 (1972) 737.
 - [2] J. D. Bekenstein, Phys. Rev. D 7 (1973) 2333.
 - [3] J. D. Bekenstein, Phys. Rev. D 9 (1974) 3292.
 - [4] S. W. Hawking, Commun. Math. Phys. 43 (1975) 199.
 - [5] J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys. 31 (1973) 161.
 - [6] F. Weinhold, J. Chem. Phys. 63 (1975) 2479;
 - G. Ruppeiner, Phys. Rev. A 20 (1979) 1608;

- G. Ruppeiner, Rev. Mod. Phys. 67 (1995) 605; 68 (1996) 313(E).
- [7] P. C. W. Davies, Proc. R. Soc. Lond. A 353 (1977) 499.
 - [8] S. Ferrara, G. W. Gibbons, and R. Kallosh, Nuc. Phys. B 500 (1997) 75;
R. G. Cai and J. H. Cho, Phys. Rev. D 60 (1999) 067502;
G. Arcioni and E. Lozano-Tellechea, Phys. Rev. D 72 (2005) 104021;
J. Aman, and N. Pidokrajt, Phys. Rev. D 73 (2006) 024017;
J. Aman, I. Bengtsson and N. Pidokrajt, Gen. Rel. Grav. 38 (2006) 1305;
H. Quevedo, J. Math. Phys. **48** (2007) 013506 [arXiv:physics/0604164];
T. Sarkar, G. Sengupta and B. N. Tiwari, JHEP 0611 (2006) 015;
J. L. Alvarez, H. Quevedo and A. Sanchez, arXiv:0801.2279 [gr-qc];
A.J.M. Medved, “A Commentary on Ruppeiner Metrics for Black Holes”, arXiv:0801.3497 [gr-qc].
 - [9] J. Aman, I. Bengtsson and N. Pidokrajt, Gen. Rel. Grav. 35 (2003) 1733.
 - [10] H. Quevedo, “Geometrothermodynamics of black holes,” arXiv:0704.3102 [gr-qc].
 - [11] J. Shen, R. G. Cai, B. Wang and R. K. Su, Int. J. Mod. Phys. A **22**, 11 (2007).
 - [12] B. Mirza and M. Zamani-Nasab, JHEP 0706 (2007) 059.
 - [13] F. Weinhold, Phys. Today **29** (1976) 23.
 - [14] G. Ruppeiner, “Thermodynamic curvature and phase transitions in Kerr-Newman black holes”, arXiv:0802.1326 [gr-qc].
 - [15] G. Ruppeiner, “Black Holes: Fermions at the Extremal Limit?”, arXiv:0711.4328 [gr-qc]
 - [16] K. S. Thorne, R. H. Price, D. A. Macdonald, *Black Holes: The Membrane Paradigm* (Yale University Press, New Haven, 1986).
 - [17] E. Witten, Phys. Rev. D 44 (1991) 314.
 - [18] G. Mandal, A. M. Sengupta, S. R. Wadia, Mod. Phys. Lett. A 6 (1991) 1685.
 - [19] C. G. Callan, S. B. Giddings, J. A. Harvey, A. Strominger, Phys. Rev. D 45 (1992) R1005.
 - [20] J. G. Russo, L. Susskind, L. Thorlacius, Phys. Lett. B 292 (1992) 13.
 - [21] V. P. Frolov, Phys. Rev. D 46 (1992) 5383.
 - [22] D. Grumiller, W. Kummer, D. V. Vassilevich, Phys. Rept. 369 (2002) 327.
 - [23] R. Jackiw, in *Quantum Theory of Gravity*, ed. S. M. Christensen (Hilger, Bristol, 1984).
 - [24] C. Teitelboim, in *Quantum Theory of Gravity*, ed. S. M. Christensen (Hilger, Bristol, 1984).
 - [25] M. Henneaux, Phys. Rev. Lett. 54 (1985) 959.

- [26] R. B. Mann, D. Robbins, and T. Ohta, Phys. Rev. Lett. 82 (1999) 3738.
- [27] A. Fabbri, D. J. Navarro, and J. Navarro-Salas, Nucl. Phys. B 595 (2001) 381.
- [28] Y.S. Myung, Y.-W. Kim and Y.-J. Park, Mod. Phys. Lett. A 23 (2008) 91;
Y.S. Myung, Y.-W. Kim and Y.-J. Park, “Thermodynamics of regular black hole”,
arXiv:0708.3145 [gr-qc];
Y.S. Myung, “Phase transition between non-extremal and extremal Reissner-Nordstrom black
holes”, arXiv:0710.2568 [gr-qc].
- [29] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, Phys. Rev. D 60 (1999) 064018.
- [30] T. Tranah and P. T. Landsberg, Collective Phenomena **3** (1980) 81
- [31] X. N. Wu, Phys. Rev. D 62 (2000) 124023.
- [32] General Relativity, Einstein & All package (GREAT.m) (2003),
<http://library.wolfram.com/infocenter/MathSource/4781/>
- [33] J. Gegenberg, G. Kunstatter and D. Louis-Martinez, Phys. Rev. D 51 (1995) 1781.
- [34] J. Cruz, A. Fabbri, D. J. Navarro, and J. Navarro-Salas, Phys. Rev. D 61 (2000) 024011.
- [35] Y. S. Myung, Y. W. Kim and Y. J. Park, Phys. Rev. D 76 (2007) 104045.
- [36] D. Grumiller and R. McNees, JHEP 0704 (2007) 074.
- [37] J. Navarro-Salas and P. Navarro, Nucl. Phys. B 579 (2000) 250.
- [38] M. Cvetič and S. S. Gubser, JHEP **9904** (1999) 024 [arXiv:hep-th/9902195]; S. S. Gubser
and I. Mitra, JHEP **0108** (2001) 018 [arXiv:hep-th/0011127]; A. Sheykhi and N. Riazi, Phys.
Rev. D **75** (2007) 024021 [arXiv:hep-th/0610085].